UNIT – II
TIME RESPONSE ANALYSIS-I

Topics: Standard test signals, Step response of first order and second order systems, Time response specifications, Time response specifications of second order systems, steady state errors and error constants. Introduction to PI, PD and PID Controllers (excluding design).

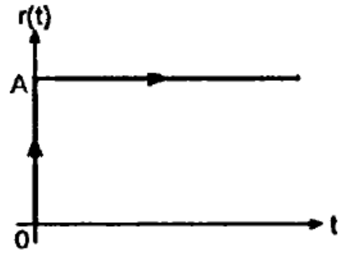
STANDARD TEST SIGNALS

i) Step Input (Position function) :

It is the sudden application of the input at a specified time as shown in the Fig.

Mathematically it can be described as,

$$\boxed{\begin{aligned} r(t) &= A \qquad \text{for } t \geq 0 \\ &= 0 \qquad \text{for } t < 0 \end{aligned}}$$

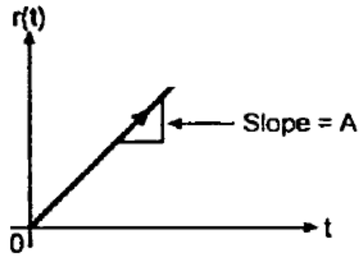


If A = 1, then it is called **unit step function** and denoted by u(t).

Laplace transform of such input is $\frac{A}{s}$.

ii) Ramp Input (Velocity function) :

It is constant rate of change in input i.e. gradual application of input as shown in the Fig.



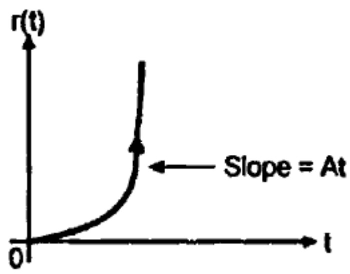
Magnitude of Ramp input is nothing but its slope. Mathematically it is defined as,

$$\boxed{\begin{aligned} r(t) &= At \qquad \text{for } t \geq 0 \\ &= 0 \qquad \text{for } t < 0 \end{aligned}}$$

If A = 1, it is called **Unit Ramp input**. It is denoted as r(t). Its Laplace transform is $\frac{A}{s^2}$.

iii) Parabolic Input (Acceleration function) :

This is the input which is one degree faster than a ramp type of input as shown in the Fig.



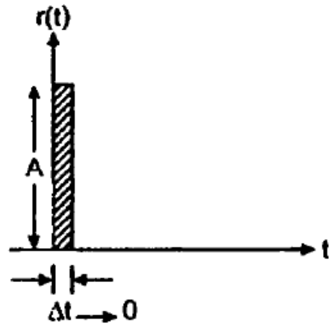
Mathematically this function is described as,

$$\boxed{\begin{aligned} r(t) &= \frac{A}{2}t^2, \qquad \text{for } t \geq 0 \\ &= 0, \qquad \text{for } t < 0 \end{aligned}}$$

where A is called magnitude of the parabolic input.

If A = 1, i.e. $r(t) = \frac{t^2}{2}$ it is called **unit parabolic input.** Its Laplace transform is $\frac{A}{s^3}$.

iv) Impulse Input :



It is the input applied instantaneously (for short duration of time) of very high amplitude as shown in the Fig.

It is the pulse whose magnitude is infinite while its width tends to zero i.e. $t \rightarrow 0$, applied momentarily .

Area of the impulse is nothing but its magnitude. If its area is unity it is called **Unit Impulse Input,** denoted as $\delta(t)$.

Mathematically it can be expressed as,

$$\therefore \quad \boxed{\begin{aligned} r(t) &= A, \quad \text{for } t = 0 \\ &= 0, \quad \text{for } t \neq 0 \end{aligned}}$$

The Laplace transform of unit impulse input is always 1. The unit impulse is denoted as $\delta(t)$.

r(t)	Symbol	R(s)
Unit step	u(t)	1/s
Unit ramp	r(t)	$1/s^2$
Unit parabolic	–	$1/s^3$
Unit impulse	$\delta(t)$	1

CHARACTERISTIC EQUATION OF A SYSTEM

Let T.F. $T(s) = \frac{C(s)}{R(s)}$

$$T(s) = \frac{b_0 s^n + b_1 s^{n-1} + \ldots\ldots + b_n}{a_0 s^n + a_1 s^{n-1} + \ldots\ldots + a_n} \quad \text{——— (1)}$$

When the denominator polynomial of transfer function is equal to zero, it is called as characteristic equation of the system.

From equation(1), characteristic equation is

$$a_0 s^n + a_1 s^{n-1} + \ldots\ldots + a_n = 0$$

ORDER OF A SYSTEM

Let T.F. $T(S) = \frac{C(S)}{R(S)}$

$$T(S) = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_n}{a_0 s^n + a_1 s^{n-1} + \ldots + a_n} \quad \text{--- (1)}$$

The order of the system is given by the max. power of 's' in the denominator polynomial of T.F.

From eq (1) ⇒ n is the order of the system.

If $n=0$, then the system is zero order system.

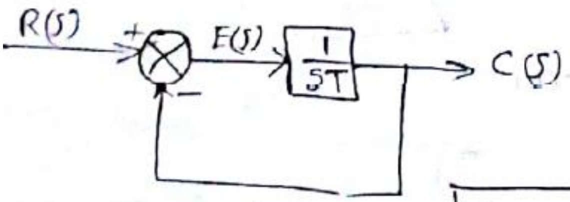
$n=1$, " " is First "

$n=2$, " " Second "

The value of n is also gives the no. of poles in T.F.

FIRST ORDER SYSTEM

Let us consider the following 1st order system with unity feedback



$$\Rightarrow \frac{C(S)}{R(S)} = \frac{1}{ST+1}$$

$$\boxed{C(S) = \frac{1}{ST+1} R(S)}$$

RESPONSE OF FIRST ORDER SYSTEM FOR UNIT STEP INPUT

For unit step $R(S) = \frac{1}{S}$

∴ From eq.(2) $C(S) = \frac{1}{ST+1} \cdot R(S)$

Sub R(S) value in the above eq.

$$C(S) = \frac{1}{S(ST+1)}$$

$$= \frac{A}{S} + \frac{B}{ST+1} = A(ST+1) + BS$$

If $s = 0$, $A = 1$

$s = -\frac{1}{T}$ $B = -T$

$$\therefore \quad C(S) = \frac{1}{S} - \frac{T}{ST+1}$$

$$= \frac{1}{S} - \frac{\cancel{T}}{\cancel{T}\left(S+\frac{1}{T}\right)}$$

$$C(S) = \frac{1}{S} - \frac{1}{S+\frac{1}{T}}$$

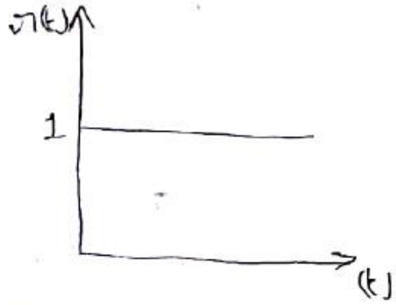
Taking inverse laplace Transform

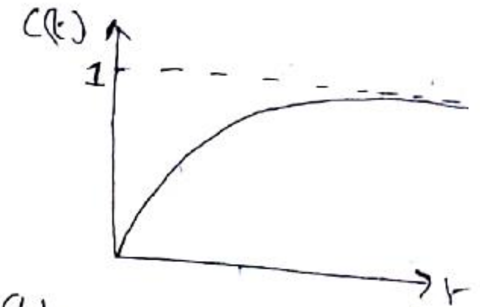
∴ The response in time domain is given by

$$\boxed{C(t) = 1 - e^{-t/T}}$$

where T = Time constant of the system.

The i/p and o/p responses are shown below.



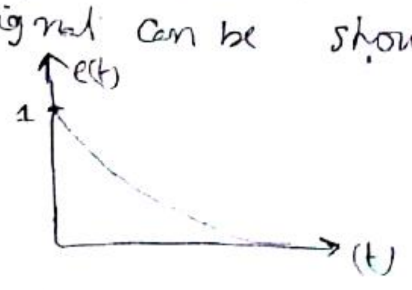


The error $e(t) = r(t) - C(t)$

$$= 1 - (1 - e^{-t/T})$$

$$= e^{-t/T}.$$

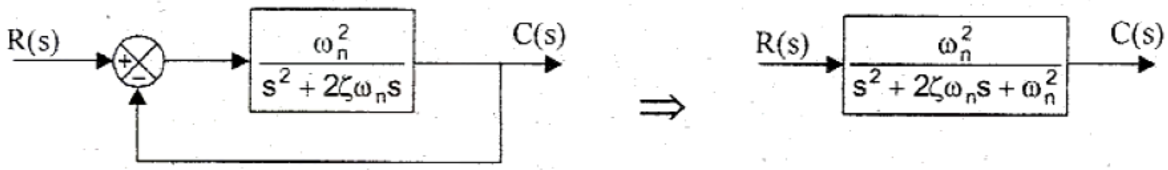
The error signal can be shown below



Steady state error $= \lim_{t \to \infty} e(t) = 0$

SECOND ORDER SYSTEM

The closed loop second order system is shown in fig



The standard form of closed loop transfer function of second order system is given by,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

where, ω_n = Undamped natural frequency, rad/sec.

ζ = Damping ratio.

*The **damping ratio*** is defined as the ratio of the actual damping to the critical damping. The response c(t) of second order system depends on the value of damping ratio. Depending on the value of ζ, the system can be classified into the following four cases,

Case 1	:	Undamped system,	$\zeta = 0$
Case 2	:	Under damped system,	$0 < \zeta < 1$
Case 3	:	Critically damped system,	$\zeta = 1$
Case 4	:	Over damped system,	$\zeta > 1$

The characteristics equation of the second order system is,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

It is a quadratic equation and the roots of this equation is given by,

$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2-1)}}{2}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

When $\zeta = 0$, $s_1, s_2 = \pm j\omega_n$; $\begin{cases}\text{roots are purely imaginary}\\ \text{and the system is undamped}\end{cases}$

When $\zeta = 1$, $s_1, s_2 = -\omega_n$; $\begin{cases}\text{roots are real and equal and}\\ \text{the system is critically damped}\end{cases}$

When $\zeta > 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$; $\begin{cases} \text{roots are real and unequal and} \\ \text{the system is overdamped} \end{cases}$

When $0 < \zeta < 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)}$

$$= -\zeta\omega_n \pm \omega_n\sqrt{-1}\,\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$= -\zeta\omega_n \pm j\omega_d \,; \begin{cases} \text{roots are complex conjugate} \\ \text{the system is underdamped} \end{cases}$$

where, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Here ω_d is called damped frequency of oscillation of the system and its unit is rad/sec.

RESPONSE OF SECOND ORDER SYSTEM FOR UNIT STEP INPUT

CASE-I : UNDAMPED SYSTEM

For undamped system, $\zeta = 0$

$$\therefore \frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + \omega_n^2}$$

For unit step, $R(S) = \frac{1}{S}$

$$\therefore C(S) = \frac{\omega_n^2}{S(S^2+\omega_n^2)}$$

$$= \frac{A}{S} + \frac{B}{S^2+\omega_n^2}$$

$$\omega_n^2 = A(S^2+\omega_n^2) + BS \quad \text{—— (1)}$$

Put $S = 0 \Rightarrow A\cancel{\omega_n^2} = \cancel{\omega_n^2}$

$$A = 1$$

Put $A = 1$ in the eq (1)

$$\cancel{\omega_n^2} = S^2 + \cancel{\omega_n^2} + B\cancel{S}$$

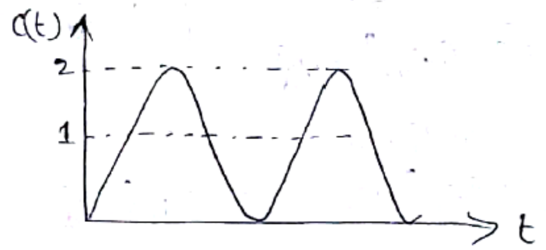
$$\therefore B = -S$$

$$\therefore C(S) = \frac{1}{S} - \frac{S}{S^2+\omega_n^2}$$

Taking inverse Laplace Transform

$$\therefore \boxed{C(t) = 1 - \cos\omega_n t}$$

The response can be shown below



∴ The response of undamped second order system for unit step i/p is completely oscillatory.

CASE-II : UNDERDAMPED SYSTEM

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For underdamped system, $0 < \zeta < 1$ and roots of the denominator (characteristic equation) are complex conjugate.

The roots of the denominator are, $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

Since $\zeta < 1$, ζ^2 is also less then 1, and so $1 - \zeta^2$ is always positive.

$$\therefore\ s = -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

The damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore s = -\zeta\omega_n \pm j\omega_d$$

The response in s-domain, $C(s) = R(s)\,\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

For unit step input, $r(t) = 1$ and $R(s) = 1/s$.

$$\therefore C(s) = \frac{\omega_n^2}{s\,(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

By partial fraction expansion, $C(s) = \dfrac{\omega_n^2}{s\,(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \dfrac{A}{s} + \dfrac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

A is obtained by multiplying C(s) by s and letting $s = 0$.

$$\therefore\ A = s \times C(s)\big|_{s=0} = s \times \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}\Bigg|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

To solve for B and C, cross multiply equation and equate like power of s.
On cross multiplication equation after substituting $A = 1$, we get,

$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + (Bs + C)s$$
$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + Bs^2 + Cs$$

Equating coefficients of s^2 we get, $0 = 1 + B$ $\quad \therefore B = -1$

Equating coefficient of s we get, $0 = 2\zeta\omega_n + C$ $\qquad \therefore C = -2\zeta\omega_n$

$$\therefore\ C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Let us add and subtract $\zeta^2\omega_n^{\ 2}$ to the denominator of second term in the equation

$$\therefore\ C(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2 + \zeta^2\omega_n^2 - \zeta^2\omega_n^2} = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2) + (\omega_n^2 - \zeta^2\omega_n^2)}$$

$$= \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} \qquad \boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$$

$$= \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

Let us multiply and divide by ω_d in the third term of the equation

$$\therefore\ C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\,\frac{\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$\mathcal{L}\{1\} = \frac{1}{s}$
$\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2 + \omega^2}$
$\mathcal{L}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2 + \omega^2}$

The response in time domain is given by,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\,\frac{\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2}\right\}$$

$$= 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}\, e^{-\zeta\omega_n t}\sin\omega_d t = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_d t\right) \qquad \boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\ \cos\omega_d t + \zeta\ \sin\omega_d t\right) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin\omega_d t \times \zeta + \cos\omega_d t \times \sqrt{1-\zeta^2}\right)$$

Let us express c(t) in a standard form as shown below.

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\sin\omega_d t \times \cos\theta + \cos\omega_d t \times \sin\theta)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)$$

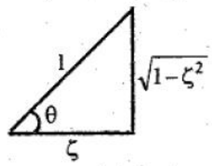
where, $\left(\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

Note : On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get

$\sin\theta = \sqrt{1-\zeta^2}$

$\cos\theta = \zeta$

$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$

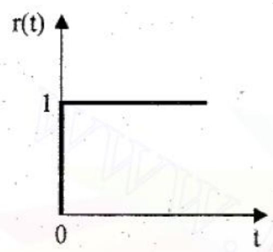


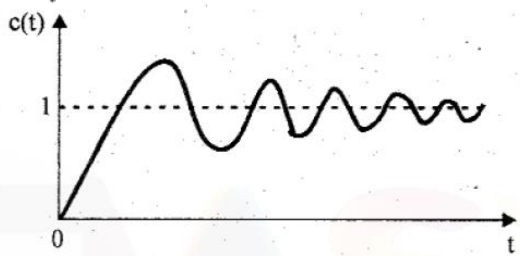
The equation is the response of under damped closed loop second order system for unit step input. For step input of step value, A, the equation should be multiplied by A.

$\therefore$ For closed loop under damped second order system,

$$\text{Unit step response} = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)\ ;\ \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\text{Step response} = A\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)\ ;\ \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right]$$





CASE-III : CRITICALLY DAMPED SYSTEM

For critically damped system, $\zeta = 1$

$$\therefore \quad \frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\omega_n S + \omega_n^2} = \frac{\omega_n^2}{(S+\omega_n)^2}$$

For step input, $R(S) = \frac{1}{S}$

$$C(S) = \frac{\omega_n^2}{S\,(S+\omega_n)^2}$$

$$= \frac{A}{S} + \frac{B}{S+\omega_n} + \frac{C}{(S+\omega_n)^2}$$

$$\omega_n^2 = A\,(S+\omega_n)^2 + B\,S\,(S+\omega_n) + C\,S$$

Put $S = 0$, $\quad A = 1$

coe. of S^2, $\quad A + B = 0$

$\quad B = -1$

coe. of S $\quad 2\omega_n A + B\omega_n + C = 0$

$\quad C = -\omega_n$

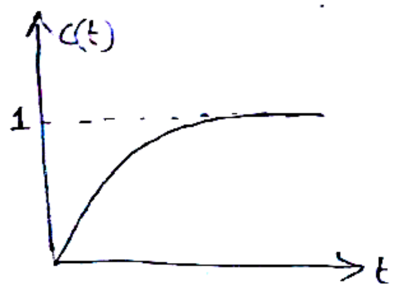
$$\therefore \quad C(S) = \frac{1}{S} - \frac{1}{S+\omega_n} - \frac{\omega_n}{(S+\omega_n)^2}$$

Taking inverse transform

$$C(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$

$$= 1 - e^{-\omega_n t}(1 + \omega_n t)$$

The response have no oscillation & exponentially increased and the response can be shown below.



CASE-IV : OVER-DAMPED SYSTEM

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

For overdamped system $\zeta > 1$. The roots of the denominator of transfer function are real and distinct. Let the roots of the denominator be s_a, s_b.

$$s_a,\ s_b=-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}=-\left[\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}\right]$$

Let $s_1=-s_2$ and $s_2=-s_b$ $\quad \therefore s_1=\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}$

$$s_2=\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}$$

The closed loop transfer function can be written in terms of s_1 and s_2 as shown below.

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{\omega_n^2}{(s+s_1)\,(s+s_2)}$$

For unit step input $r(t) = 1$ and $R(s) = 1/s$.

$$\therefore C(s)=R(s)\frac{\omega_n^2}{(s+s_1)\,(s+s_2)}=\frac{\omega_n^2}{s\,(s+s_1)\,(s+s_2)}$$

By partial fraction expansion we can write,

$$C(s)=\frac{\omega_n^2}{s\,(s+s_1)\,(s+s_2)}=\frac{A}{s}+\frac{B}{s+s_1}+\frac{C}{s+s_2}$$

$$A=s\times C(s)\big|_{s=0}=s\times\frac{\omega_n^2}{s\,(s+s_1)\,(s+s_2)}\bigg|_{s=0}=\frac{\omega_n^2}{s_1 s_2}$$

$$=\frac{\omega_n^2}{\left[\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right]\left[\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right]}=\frac{\omega_n^2}{\zeta^2\omega_n^2-\omega_n^2(\zeta^2-1)}=\frac{\omega_n^2}{\omega_n^2}=1$$

$$B=(s+s_1)\times C(s)\big|_{s=-s_1}=\frac{\omega_n^2}{s(s+s_2)}\bigg|_{s=-s_1}=\frac{\omega_n^2}{-s_1(-s_1+s_2)}$$

$$=\frac{-\omega_n^2}{s_1\left[-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right]}=\frac{-\omega_n^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]s_1}=\frac{-\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}$$

$$C=C(s)\times(s+s_2)\big|_{s=-s_2}=\frac{\omega_n^2}{s\,(s+s_1)}\bigg|_{s=-s_2}=\frac{\omega_n^2}{-s_2(-s_2+s_1)}$$

$$=\frac{\omega_n^2}{-s_2\left[-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right]}=\frac{\omega_n^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]s_2}=\frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_2}$$

The response in time domain, c(t) is given by,

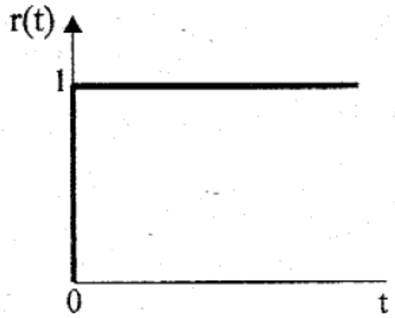
$$c(t)=\mathcal{L}^{-1}\left\{=\frac{1}{s}-\frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}\,\frac{1}{(s+s_1)}+\frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_2}\,\frac{1}{(s+s_2)}\right\}$$

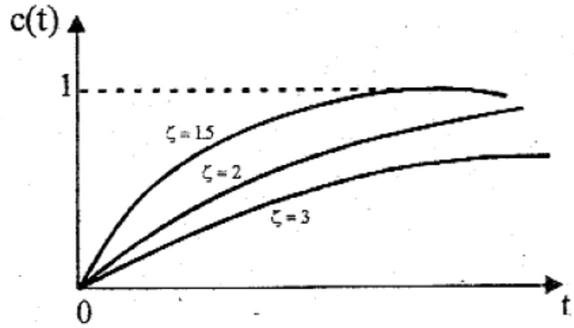
$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\ \frac{1}{s_1}\ e^{-s_1 t} + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\ \frac{1}{s_2}\ e^{-s_2 t}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2} \right)$$

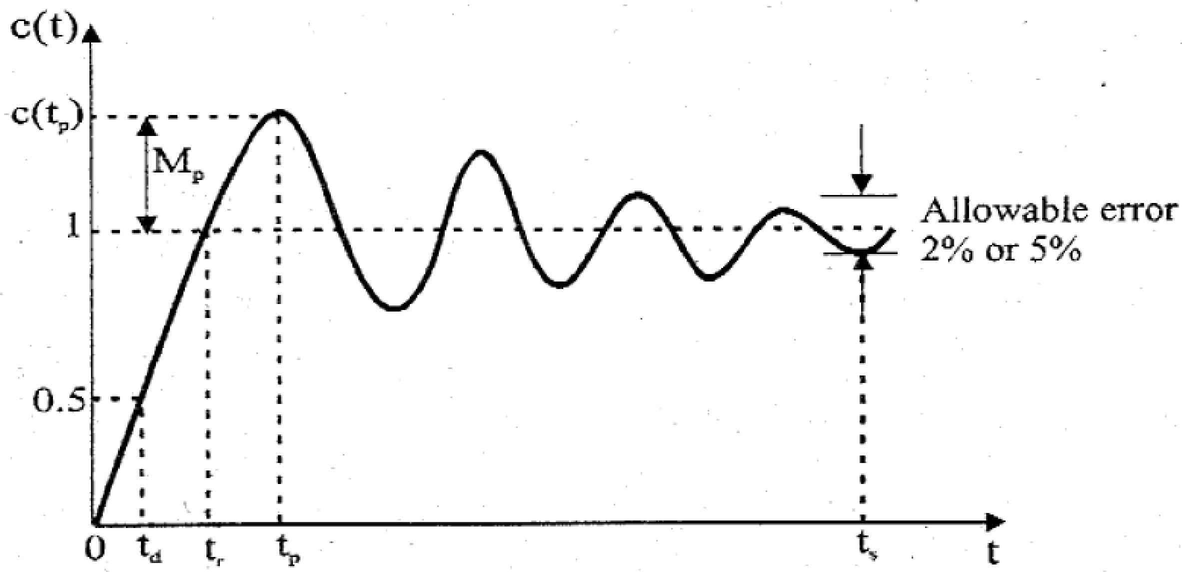
$$\text{where, } s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$





TIME RESPONSE SPECIFICATIONS OF SECOND ORDER SYSTEM



1. Delay time, t_d
2. Rise time, t_r
3. Peak time, t_p
4. Maximum overshoot, M_p
5. Settling time, t_s

1. Delay time (t_d): It is the time required for the response to reach 50% of the steady state value for the first time

2. Rise time (t_r): It is the time required for the response to reach 100% of the steady state value for under damped systems. However, for over damped systems, it is taken as the time required for the response to rise from 10% to 90% of the steady state value.

3. Peak time (t_p): It is the time required for the response to reach the maximum or Peak value of the response.

4. Peak overshoot (M_p) : It is defined as the difference between the peak value of the response and the steady state value. It is usually expressed in percent of the steady state value. If the time for the peak is t_p, percent peak overshoot is given by,

$$\text{Percent peak overshoot } M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100.$$

For systems of type 1 and higher, the steady state value c (∞) is equal to unity, the same as the input.

5. Settling time (t_s) : It is the time required for the response to reach and remain within a specified tolerance limits (usually ± 2% or ± 5%) around the steady state value.

EXPRESSION FOR RISE TIME (t_r)

The unit step response of second order system for underdamped case is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

At $t = t_r$, $c(t) = c(t_r) = 1$

$$\therefore c(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 1$$

$$\therefore \frac{-e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 0$$

Since $-e^{-\zeta\omega_n t_r} \neq 0$, the term, $\sin(\omega_d t_r + \theta) = 0$

When, $\phi = 0, \pi, 2\pi, 3\pi\ldots,$ $\quad \sin\phi = 0$

$$\therefore \omega_d t_r + \theta = \pi$$

$$\omega_d t_r = \pi - \theta$$

$$\therefore \text{Rise Time, } t_r = \frac{\pi - \theta}{\omega_d}$$

Here, $\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$; Damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore \text{Rise time, } t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}} \text{ in sec}$$

EXPRESSION FOR PEAK TIME (t_p)

To find the expression for peak time, t_p, differentiate c(t) with respect to t and equate to 0.

i.e., $\frac{d}{dt}c(t)\Big|_{t=t_p} = 0$

The unit step response of under damped second order system is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)$$

Differentiating c(t) with respect to t.

$$\frac{d}{dt}c(t) = \frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(-\zeta\omega_n)\sin(\omega_d t + \theta) + \left(\frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\cos(\omega_d t + \theta)\omega_d$$

Put, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore \frac{d}{dt}c(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\zeta\omega_n)\sin(\omega_d t + \theta) - \frac{\omega_n\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_d t + \theta)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\zeta\sin(\omega_d t + \theta) - \sqrt{1-\zeta^2}\cos(\omega_d t + \theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\cos\theta\sin(\omega_d t + \theta) - \sin\theta\cos(\omega_d t + \theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\sin(\omega_d t + \theta)\cos\theta - \cos(\omega_d t + \theta)\sin\theta\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\sin((\omega_d t + \theta) - \theta)\right] = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t)$$

at $t = t_p$, $\frac{d}{dt}c(t) = 0$

$$\therefore \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p}\sin(\omega_d t_p) = 0$$

Since, $e^{-\zeta\omega_n t_p} \neq 0$, the term, $\sin(\omega_d t_p) = 0$

When $\phi = 0, \pi, 2\pi, 3\pi$, $\sin\phi = 0$

$\therefore \ \omega_d t_p = \pi$

$$\therefore \text{Peak time, } t_p = \frac{\pi}{\omega_d}$$

The damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\boxed{\therefore \text{Peak time, } t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$$

EXPRESSION FOR PEAK OVERSHOOT (M_p)

$$\boxed{\%\text{Peak overshoot, } \%M_p = \frac{c(t_p)-c(\infty)}{c(\infty)} \times 100}$$

where, $c(t_p)$ = Peak response at $t = t_p$.
$c(\infty)$ = Final steady state value.

The unit step response of second order system is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

$$\text{At } t=\infty, \quad c(t) = c(\infty) = 1 - \frac{e^{-\infty}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta) = 1 - 0 = 1$$

$$\text{At } t=t_p, \quad c(t) = c(t_p) = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \theta)$$

$$= 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d \frac{\pi}{\omega_d} + \theta\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} \sin(\pi + \theta)}{\sqrt{1-\zeta^2}}$$

$$= 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} \sin(\pi + \theta)}{\sqrt{1-\zeta^2}}$$

$$= 1 + \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\theta$$

$$= 1 + \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sqrt{1-\zeta^2} = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$\text{Percentage Peak Overshoot, } \%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100 = \frac{1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} - 1}{1} \times 100$$

$$= e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

EXPRESSION FOR SETTLING TIME (t_s)

The response of second order system has two components. They are,

1. Decaying exponential component, $\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$.
2. Sinusoidal component, $\sin(\omega_d t + \theta)$.

In this the decaying exponential term dampens (or) reduces the oscillations produced by sinusoidal component. Hence the settling time is decided by the exponential component. The settling time can be found out by equating exponential component to percentage tolerance errors.

For 2 % tolerance error band, at $t = t_s$, $\frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.02$

For least values of ζ , $e^{-\zeta\omega_n t_s} = 0.02$.

On taking natural logarithm we get,

$$-\zeta\omega_n t_s = ln(0.02) \quad \Rightarrow \quad -\zeta\omega_n t_s = -4 \quad \Rightarrow \quad t_s = \frac{4}{\zeta\omega_n}$$

For the second order system, the time constant, $T = \frac{1}{\zeta\omega_n}$

$$\boxed{\therefore \text{ Settling time, } t_s = \frac{1}{\zeta\omega_n} = 4T \qquad \text{(for 2\% error)}}$$

For 5% error, $e^{-\zeta\omega_n t_s} = 0.05$

On taking natural logarithm we get,

$$-\zeta\omega_n t_s = ln(0.05) \quad \Rightarrow \quad -\zeta\omega_n t_s = -3 \quad \Rightarrow \quad t_s = \frac{3}{\zeta\omega_n}$$

$$\boxed{\therefore \text{ Settling time, } t_s = \frac{3}{\zeta\omega_n} = 3T \qquad \text{(for 5\% error)}}$$

$$\boxed{\therefore \text{ Settling time, } t_s = \frac{ln(\%\text{ error})}{\zeta\omega_n} = \frac{ln(\%\text{ error})}{T}}$$

EXPRESSION FOR DELAY TIME (t_d)

Delay time is the time required to reach 50% of o/p.

i.e. $c(t_d) = \frac{1}{2}$

The unit step response of second order system is given by

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

At $t = t_d$, $c(t) = \frac{1}{2}$

$$\therefore \frac{1}{2} = 1 - \frac{e^{-\xi\omega_n t_d}}{\sqrt{1-\xi^2}} \sin(\omega_d t_d + \theta)$$

By solving this eq.

$$\boxed{t_d = \frac{1+0.7\xi}{\omega_n}}$$

ZERO'S OF THE SYSTEM

The roots of numerator polynomial of the transfer function are called as zero's of the system. (Or)

The values of 's' at which the transfer function becomes zero are called as zero's of the system.

POLE OF THE SYSTEM

The roots of denominator polynomial of the transfer function are called as poles of the system. (Or)

The values of 's' at which the transfer function becomes infinity are called as poles of the system.

TYPE NUMBER OF SYSTEM

The type no. is specified for loop T.F $G(S)H(S)$.

The type number is defined as the no. of poles of the loop T.F. lying at origin.

If N is the no. of poles at origin then type no. is N.

Consider

$$G(S)H(S) = \frac{K(S+z_1)(S+z_2)\ldots\ldots}{S^N(S+P_1)(S+P_2)\ldots\ldots}$$

where $z_1, z_2 \ldots$ = zeros of T.F

$P_1, P_2 \ldots\ldots$ = poles ,,

K = constant.

N = no. of poles at origin

= Type No. of system.

If $N=0$, then the system is type-0 system.

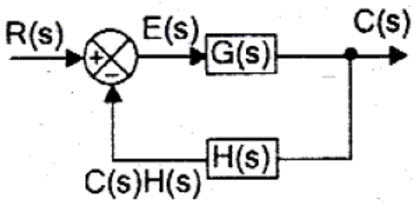
$N=1$, ,, ,, type-1 ,,

$N=2$,, ,, type-2 ,, so on

STEADY STATE ERROR

The steady state error is the value of error signal e(t), when **t** tends to infinity. The steady state error is a measure of system accuracy. These errors arise from the nature of inputs, type of system and from non linearity of system components. The steady state performance of a stable control system is generally judged by its steady state error to step, ramp and parabolic inputs.

Consider a closed loop system shown in fig



Let, R(s) = Input signal

E(s) = Error signal

C(s) H(s) = Feedback signal

C(s) = Output signal or response

The error signal, E(s) = R(s) – C(s) H(s)

The output signal, C(s) = E(s) G(s)

On substituting for C(s)

$$E(s) = R(s) - [E(s)\ G(s)]\ H(s)$$

$$E(s) + E(s)\ G(s)\ H(s) = R(s)$$

$$E(s)\ [1 + G(s)\ H(s)] = R(s)$$

$$\therefore E(s) = \frac{R(s)}{1+G(s)\ H(s)}$$

Let, e(t) = error signal in time domain.

$$\therefore e(t) = \mathcal{L}^{-1}\{E(s)\} = \mathcal{L}^{-1}\left\{\frac{R(s)}{1+G(s)\ H(s)}\right\}$$

Let, e_{ss} = steady state error.

The steady state error is defined as the value of e(t) when **t** tends to infinity.

$$\therefore e_{ss} = \underset{t\to\infty}{\mathrm{Lt}}\ e(t)$$

The final value theorem of Laplace transform states that,

$$\textbf{If,}\quad \mathbf{F(s) = \mathcal{L}\{f(t)\}}\ \textbf{then,}\quad \underset{t\to\infty}{\mathbf{Lt}}\ \mathbf{f(t)} = \underset{s\to 0}{\mathbf{Lt}}\ \mathbf{s\,F(s)}$$

Using final value theorem,

$$\text{The steady state error, } e_{ss} = \underset{t\to\infty}{\mathrm{Lt}}\ e(t) = \underset{s\to 0}{\mathrm{Lt}}\ sE(s) = \underset{s\to 0}{\mathrm{Lt}}\ \frac{sR(s)}{1+G(s)\ H(s)}$$

STATIC ERROR CONSTANTS

When a control system is excited with standard input signal, the steady state error may be zero, constant or infinity. The value of steady state error depends on the type number and the input signal. Type-0 system will have a constant steady state error when the input is step signal. Type-1 system will have a constant steady state error when the input is ramp signal or velocity signal. Type-2 system will have a constant steady state error when the input is parabolic signal or acceleration signal. For the three cases mentioned above the steady state error is associated with one of the constants defined as follows,

$$\text{Positional error constant, } K_p = \underset{s\to 0}{\mathrm{Lt}}\ G(s)\ H(s)$$

$$\text{Velocity error constant, } K_v = \underset{s\to 0}{\mathrm{Lt}}\ s\ G(s)\ H(s)$$

$$\text{Acceleration error constant, } K_a = \underset{s\to 0}{\mathrm{Lt}}\ s^2 G(s)\ H(s)$$

The K_p, K_v and K_a are in general called static error constants.

STEADY STATE ERROR FOR UNIT STEP SIGNAL

Steady state error $e_{ss} = \lim_{s \to 0} \frac{s\,R(s)}{1+G(s)H(s)}$

For unit step, $R(s) = \frac{1}{s}$

$$\therefore\ e_{ss} = \lim_{s \to 0} \frac{1}{1+G(s)H(s)}$$

$$= \frac{1}{1+\lim_{s \to 0} G(s)H(s)}$$

$$\boxed{e_{ss} = \frac{1}{1+K_p}}$$

For Type-0 System :- $K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \frac{K(s+z_1)(s+z_2)\ldots}{(s+p_1)(s+p_2)\ldots}$

$$K_p = \text{const.}$$

$$\therefore\ e_{ss} = \frac{1}{1+K_p} = \text{const.}$$

Hence In type-0 system when the i/p is unit step, steady state error is constant.

For Type-1 system :- $K_p = \lim_{s \to 0} \frac{K(s+z_1)(s+z_2)\ldots}{s(s+p_1)(s+p_2)\ldots}$

$$K_p = \infty$$

$$\therefore\ e_{ss} = \frac{1}{1+\infty} = 0.$$

Hence in type-1 systems when the i/p is unit step, steady state error is zero.

STEADY STATE ERROR FOR UNIT RAMP SIGNAL

steady state error $e_{ss} = \lim_{s \to 0} \frac{s R(s)}{1 + G(s) H(s)}$

For unit Ramp $R(s) = \frac{1}{s^2}$

$$\therefore e_{ss} = \lim_{s \to 0} \frac{\cancel{s} \cdot 1/s^{\cancel{2}}}{1 + G(s) H(s)}$$

$$= \lim_{s \to 0} \frac{1}{s + s G(s) H(s)}$$

$$e_{ss} = \frac{1}{\lim_{s \to 0} s G(s) H(s)}$$

$$\boxed{e_{ss} = \frac{1}{K_v}}$$

For Type - 0 system :-

$$K_v = \lim_{s \to 0} s G(s) H(s) = \lim_{s \to 0} s \frac{k (s+z_1)(s+z_2) \ldots}{(s+P_1)(s+P_2) \ldots}$$

$K_v = 0$

$$\therefore e_{ss} = \frac{1}{0} = \infty$$

For Type - 1 system :-

$$K_v = \lim_{s \to 0} \cancel{s} \frac{k (s+z_1)(s+z_2) \ldots}{\cancel{s}(s+P_1)(s+P_2) \ldots}$$

$= \text{Const.}$

$\therefore e_{ss} = \text{Const.}$

Hence steady state error is constant in type-1 system for unit ramp i/p.

For Type - 2 system :-

$$K_v = \lim_{s \to 0} \cancel{s} \frac{K (s+z_1)(s+z_2) \ldots}{s^{\cancel{2}}(s+P_1)(s+P_2) \ldots}$$

$= \infty$

$\therefore e_{ss} = 0$

Hence steady state error is zero in type - 2 system for unit ramp input.

STEADY STATE ERROR FOR UNIT PARABOLIC SIGNAL

Steady state error $e_{ss} = \lim_{s \to 0} \frac{s R(s)}{1 + G(s) H(s)}$

For unit Parabola, $R(s) = \frac{1}{s^3}$

$$\therefore e_{ss} = \frac{1}{\lim_{s \to 0} s^2 G(s) H(s)}$$

$$\boxed{e_{ss} = \frac{1}{K_a}}$$

Type-0 system :-

$$K_a = \lim_{s \to 0} s^2 G(s) H(s) = \lim_{s \to 0} s^2 \frac{K (s+z_1)(s+z_2) \ldots}{(s+P_1)(s+P_2) \ldots}$$

$$= 0$$

$$e_{ss} = \infty$$

Type-1 system :-

$$K_a = 0$$

$$e_{ss} = \infty$$

Type-2 system :-

$$K_a = \lim_{s \to 0} \frac{\cancel{s^2} K (s+z_1)(s+z_1) \ldots}{\cancel{s^2}(s+P_1)(s+P_2) \ldots}$$

$$= \text{Const}$$

$$e_{ss} = \text{Const.}$$

Type-3 system :-

$$K_a = \lim_{s \to 0} \frac{\cancel{s^2} K (s+z_1)(s+z_2)}{s^{\cancel{3}} (s+P_1)(s+P_2) \ldots}$$

$$= \infty$$

$$e_{ss} = 0$$

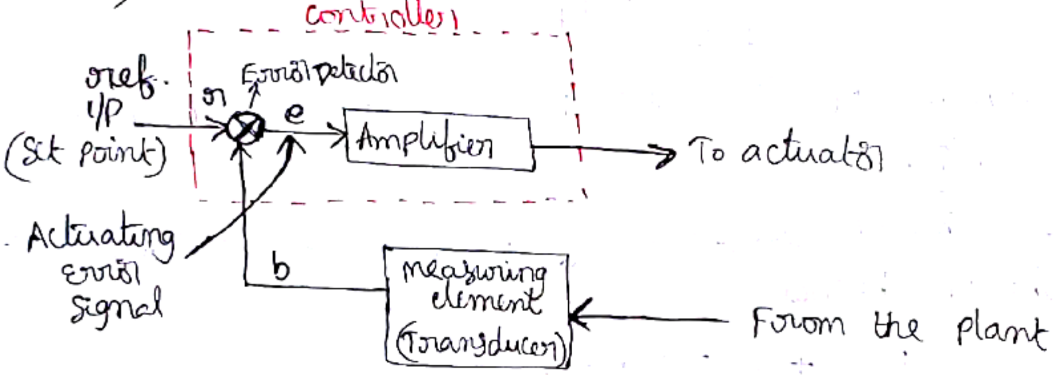
Error Constant	Type number of system			
	0	1	2	3
K_p	constant	∞	∞	∞
K_v	0	constant	∞	∞
K_a	0	0	constant	∞

Input Signal	Type number of system			
	0	1	2	3
Unit Step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0

CONTROLLERS

The automatic controller determines the value of controlled variable by comparing the actual value to the desired value (ref. I/P) i.e. the controller determines the deviation and produces a control signal that will reduce the deviation to zero or to a smallest possible value. This method of producing the control signal is called control action.

The general block diagram of automatic controller (Industrial) can be shown below.



The controller consists of an error detector and amplifier. The measuring element is a device which converts the o/p variable to another suitable variable. The set point of controller must be converted to ref I/P of the same units as feedback signal from the measuring element.

From the fig, $e = r - b$

CLASSIFICATION OF CONTROLLERS

The controllers are classified depending upon the type of controlling action used.

(i) Proportional controllers (P- controllers)
(ii) Integral controllers (I- controllers)
(iii) Derivative controllers (D- controllers)

(iv) Proportional - plus - Integral Controllers (PI controllers)
(v) Proportional - plus - Derivative controllers (PD controllers)
(vi) Proportional - plus - Integral - plus - Derivative controllers (PID)

PROPORTIONAL CONTROLLER (P-CONTROLLER)

The Proportional controller produces an o/p signal $(m(t))$ which is proportional to the error signal. $e(t)$.

i.e. $m(t) = K_p\, e(t)$

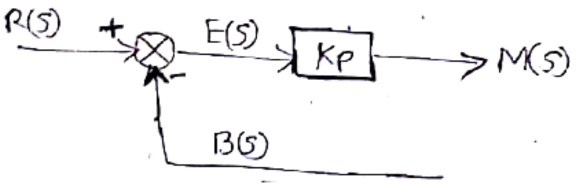
Taking Laplace Transform

$\Rightarrow M(S) = K_p E(S)$

$$T.F. = \frac{M(S)}{E(S)} = K_p$$

Where K_p is called Proportional gain / Proportional sensitivity.

The block diagram of Proportional controller can be shown below.



ADVANTAGES:

1. Proportional controller helps in reducing the steady state error, thus makes the system more stable.
2. Slow response of the over damped system can be made faster with the help of these controllers.

DISADVANTAGES:

1. Due to presence of these controllers we get some offsets in the system.
2. Proportional controllers also increases the maximum overshoot of the system.

INTEGRAL CONTROLLER (I-CONTROLLER)

The integral controller produces an o/p signal which is integral of the error signal e(t)

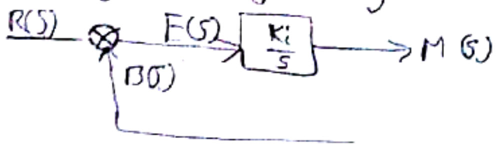
i.e. $m(t) = K_i \int e(t)$

Taking laplace Transform

$$M(s) = K_i \frac{E(s)}{s}$$

T.F $\frac{M(s)}{E(s)} = \frac{K_i}{s}$

The block diagram of integral controller is shown as



Advantages of Integral Controller:

Due to their unique ability, they can return the controlled variable back to the exact set point following a disturbance that's why these are known as reset controllers.

Disadvantages of Integral Controller:

It tends to make the system unstable because it responds slowly towards the produced error.

DERIVATIVE CONTROLLER (D-CONTROLLER)

The derivative controller produces the o/p signal m(t) which is the derivative of error signal i.e. rate of change of error signal, e(t)

i.e. $m(t) = K_d \frac{d}{dt} e(t)$ —— (1)

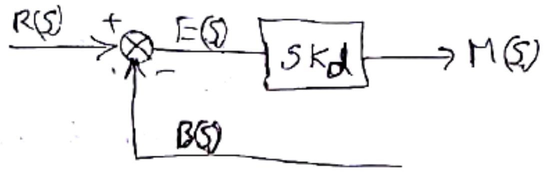
where K_d = derivative gain constant.

Taking laplace Transform

$$M(s) = K_d \, s \, E(s).$$

$$T.F = \frac{M(s)}{E(s)} = K_d s$$

The block diagram of integral controller can be shown as



From eq. ①, when the error is zero or constant, the o/p of the controller will be zero. ∴ This type of controller can't be used alone. This controller having small gain.

Advantages of Derivative Controller:
The major advantage of derivative controller is that it improves the transient response of the system.
Disadvantages of Derivative Controller:
1. It never improves the steady state error.
2. It produces saturation effects and also amplifies the noise signals produced in the system.

PROPORTIONAL+INTEGRAL CONTROLLER (PI-CONTROLLER)

This is combination of proportional and integral control action.

i.e. $m(t) = K_p e(t) + K_p K_i \int_0^t e(t)\,dt$

Integral gain $K_i = \frac{1}{T_i}$

where T_i = Time constant of Integral controller

$$\therefore m(t) = K_p e(t) + \frac{K_p}{T_i} \int e(t)\,dt$$

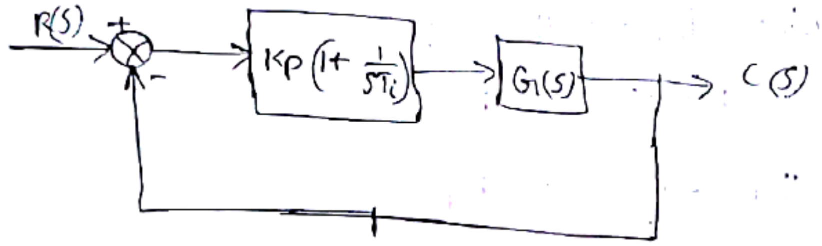
Taking laplace Transform

$$M(S) = K_p E(S) + \frac{K_p}{T_i}\frac{E(S)}{S}$$

$$= E(S)\, K_p\left(1+\frac{1}{ST_i}\right)$$

$$T.F = \frac{M(S)}{E(S)} = K_p\left(1+\frac{1}{ST_i}\right)$$

The block diagram of unity feedback system with PI controller can be shown below.



Let $G(S) = \dfrac{\omega_n^2}{S(S+2\zeta\omega_n)}$

$$\frac{C(S)}{R(S)} = \frac{K_p\left(1+\frac{1}{ST_i}\right)\frac{\omega_n^2}{S(S+2\zeta\omega_n)}}{1+K_p\left(1+\frac{1}{ST_i}\right)\frac{\omega_n^2}{S(S+2\zeta\omega_n)}}$$

$$\frac{C(S)}{R(S)} = \frac{K_p\omega_n^2(1+T_iS)}{S^2T_i(S+2\zeta\omega_n)+K_p\omega_n^2(1+T_iS)}$$

$$= \frac{(K_p/T_i)\,\omega_n^2(1+T_iS)}{S^3+2\zeta\omega_n S^2+K_p\omega_n^2 S+\frac{K_p}{T_i}\omega_n^2}$$

Let $K_i = \dfrac{K_p}{T_i}$

$$\frac{C(S)}{R(S)} = \frac{K_i\,\omega_n^2(1+T_iS)}{S^3+2\zeta\omega_n S^2+K_p\omega_n^2 S+K_i\omega_n^2}$$

Effects of PI Controller :-

1) System behaviour is accurate
2) Order and type of the system can be increased by one.
3) Damping is improved and overshoot is reduced.
4) Band width also increases
5) steady state error can be reduced
6) It adds a zero to the system

PROPORTIONAL+DERIVATIVE CONTROLLER (PD-CONTROLLER)

This controller is the combination of proportional and derivative control action.

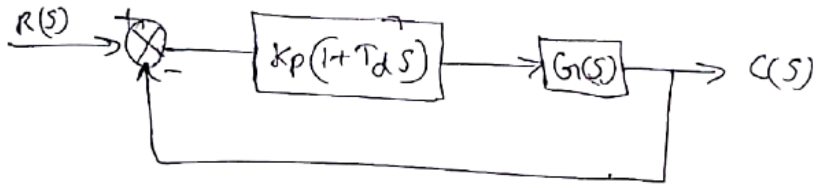
i.e. $$m(t) = K_p\, e(t) + K_p T_d \frac{d}{dt} e(t)$$

Taking laplace Transform

$$M(s) = K_p E(s) + K_p T_d\, s\, E(s)$$

$$\Rightarrow \frac{M(s)}{E(s)} = K_p(1+T_d s) = T.F.$$

The block diagram can be shown below with PD controller



let the open loop T.F, $G(s) = \dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$

$$\therefore \frac{C(s)}{R(s)} = \frac{K_p(1+T_d s)\, G(s)}{1+K_p(1+T_d s)G(s)}$$

$$= \frac{K_p(1+T_d s)\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}{1+K_p(1+T_d s)\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}$$

$$\frac{C(s)}{R(s)} = \frac{K_p \omega_n^2 (1 + T_d s)}{s^2 + (2\zeta\omega_n + K T_d \omega_n^2)s + K_p \omega_n^2}$$

Effects of PD Controller :-

1) It adds a zero to the system
2) Damping ratio improves and max. overshoot reduces.
3) Rise time & settling time are reduced.
4) Band width increases
5) Noise is filtered out.

PROPORTIONAL+INTEGRAL+DERIVATIVE CONTROLLER(PID-CONTROLLER)

The PID-controller produces an output signal consisting of three terms : *one proportional to error signal, another one proportional to integral of error signal and the third one proportional to derivative of error signal.*

$$\text{In PID - controller,}\quad u(t) \;\alpha\; \left[e(t) + \int e(t)\,dt + \frac{d}{dt}e(t)\right]$$

$$\therefore u(t) = K_p\, e(t) + \frac{K_p}{T_i}\int e(t)dt + K_p T_d \frac{d}{dt}e(t)$$

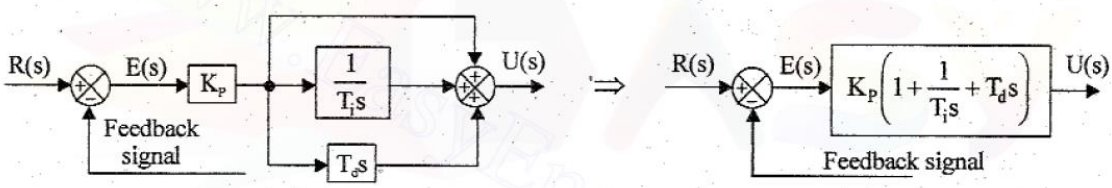
where, K_p = Proportional gain
T_i = Integral time
T_d = Derivative time

On taking Laplace transform of equation with zero initial conditions we get,

$$U(s) = K_p E(s) + \frac{K_p}{T_i}\frac{E(s)}{s} + K_p\, T_d\, s\, E(s)$$

$$\boxed{\therefore \text{ Transfer function of PID - controller,}\quad \frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)}$$

The equation gives the output of the PID-controller for the input E(s) and equation is the transfer function of the PID-controller. The block diagram of PID-contoller is shown in fig



The combination of proportional control action, integral control action and derivative control action is called PID-control action. This combined action has the advantages of the each of the three individual control actions.

The proportional controller stabilizes the gain but produces a steady state error. The integral controller reduces or eliminates the steady state error. The derivative controller reduces the rate of change of error.

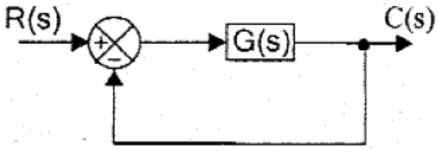
PROBLEMS

1) Obtain the response of unity feedback system when the input to the system is unit step and whose open loop transfer function is

$$G(s) = \frac{4}{s(s+5)}$$

SOL:

The closed loop system is shown in fig



The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{4}{s\,(s+5)}}{1+\frac{4}{s\,(s+5)}} = \frac{\frac{4}{s\,(s+5)}}{\frac{s\,(s+5)+4}{s\,(s+5)}} = \frac{4}{s\,(s+5)+4} = \frac{4}{s^2+5s+4} = \frac{4}{(s+4)\,(s+1)}$$

The response in s-domain, $C(s) = R(s)\,\frac{4}{(s+1)\,(s+4)}$

Since the input is unit step, $R(s) = \frac{1}{s}$; $\quad \therefore C(s) = \frac{4}{s\,(s+1)\,(s+4)}$

By partial fraction expansion, we can write,

$$C(s) = \frac{4}{s\,(s+1)\,(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$A = C(s) \times s\Big|_{s=0} = \left.\frac{4}{(s+1)\,(s+4)}\right|_{s=0} = \frac{4}{1\times 4} = 1$$

$$B = C(s) \times (s+1)\Big|_{s=-1} = \left.\frac{4}{s\,(s+4)}\right|_{s=-1} = \frac{4}{-1(-1+4)} = \frac{-4}{3}$$

$$C = C(s) \times (s+4)\Big|_{s=-4} = \left.\frac{4}{s\,(s+1)}\right|_{s=-4} = \frac{4}{-4(-4+1)} = \frac{1}{3}$$

The time domain response c(t) is obtained by taking inverse Laplace transform of C(s).

Response in time domain, $c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{4}{3}\,\frac{1}{s+1} + \frac{1}{3}\,\frac{1}{s+4}\right\}$

$$= 1 - \frac{4}{3}\,e^{-t} + \frac{1}{3}\,e^{-4t} = 1 - \frac{1}{3}\left[4e^{-t} - e^{-4t}\right]$$

2) The response of a servomechanism is $c(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$ when subject to a unit step input. Obtain an expression for closed loop transfer function. Determine the undamped natural frequency and damping ratio.

SOL:

Given that, $c(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$

On taking Laplace transform of c(t) we get,

$$C(s) = \frac{1}{s} + 0.2\,\frac{1}{(s+60)} - 1.2\,\frac{1}{(s+10)} = \frac{(s+60)\,(s+10) + 0.2\,s\,(s+10) - 1.2\,s\,(s+60)}{s\,(s+60)\,(s+10)}$$

$$= \frac{s^2 + 70s + 600 + 0.2\,s^2 + 2s - 12s^2 - 72s}{s\,(s+60)\,(s+10)} = \frac{600}{s\,(s+60)\,(s+10)} = \frac{1}{s}\,\frac{600}{(s+60)\,(s+10)}$$

Since input is unit step, $R(s) = 1/s$.

$$\therefore\ C(s) = R(s)\,\frac{60}{(s+60)\,(s+10)} = R(s)\,\frac{600}{s^2 + 70s + 600}$$

$\therefore$ The closed loop transfer function of the system, $\dfrac{C(s)}{R(s)} = \dfrac{600}{s^2 + 70s + 600}$

The damping ratio and natural frequency of oscillation can be estimated by comparing the system transfer function with standard form of second order transfer function.

$$\therefore\ \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{600}{s^2 + 70s + 600}$$

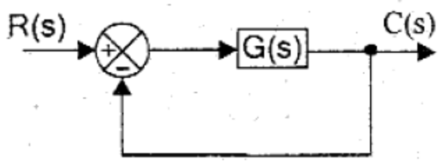
On comparing we get,

$\omega_n^2 = 600$	$2\zeta\omega_n = 70$
$\therefore\ \omega_n = \sqrt{600} = 24.49$ rad / sec	$\therefore\ \zeta = \dfrac{70}{2\omega_n} = \dfrac{70}{2 \times 24.49} = 1.43$

3) The unity feedback is characterized by an open loop transfer function G(s) = K / s(s+10). Determine the gain K so that the system will have the damping ratio of 0.5 for this value of K. Determine the peak overshoot and the time at peak overshoot for a unit step input.

SOL:

The unity feedback system is shown in fig



The closed loop transfer function $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1 + G(s)}$

Given that, $G(s) = K/s\,(s+10)$

$$\therefore\ \frac{C(s)}{R(s)} = \frac{\dfrac{K}{s(s+10)}}{1 + \dfrac{K}{s(s+10)}} = \frac{K}{s\,(s+10) + K} = \frac{K}{s^2 + 10s + K}$$

The value of K can be evaluated by comparing the system transfer function with standard form of second order transfer function.

$$\therefore\ \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K}{s^2 + 10s + K}$$

On comparing we get,

$\omega_n^2 = K$	$2\zeta\omega_n = 10$	$K = 100$
$\therefore\ \omega_n = \sqrt{K}$	Put $\zeta = 0.5$ and $\omega_n = \sqrt{K}$	$\omega_n = 10$ rad / sec
	$\therefore\ 2 \times 0.5 \times \sqrt{K} = 10$	
	$\sqrt{K} = 10$	

The value of gain, K=100.

Percentage peak overshoot, $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$

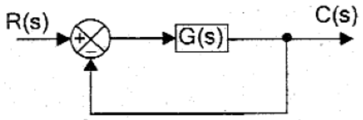
$$= e^{-0.5\pi/\sqrt{1-0.5^2}} \times 100 = 0.163 \times 100 = 16.3\%$$

Peak time, $t_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{\pi}{10\sqrt{1-0.5^2}} = 0.363$ sec

4) The open loop transfer function of a unity feedback system is G(s) = K / s(sT+1) where K and T are positive constants. By what factor should the amplifier gain K be reduced, so that the peak overshoot of unit step response of the system is reduced from 75% to 25%.

SOL:

The unity feedback system is shown in fig



The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

Given that, G(s) = K/s (sT +1)

$$\therefore \frac{C(s)}{R(s)} = \frac{K/s\,(sT+1)}{1+K/s(sT+1)} = \frac{K}{s\,(sT+1)+K} = \frac{K}{s^2T+s+K} = \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}}$$

Expression for ζ and ω_n can be obtained by comparing the transfer function with the standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}}$$

On comparing we get,

$$\omega_n^2 = K/T$$

$$\therefore \omega_n = \sqrt{K/T}$$

$$2\zeta\omega_n = 1/T$$

$$\zeta = \frac{1}{2\omega_n T} = \frac{1}{2\sqrt{\frac{K}{T}}\,T} = \frac{1}{2\sqrt{KT}}$$

The peak overshoot, M_p is reduced by increasing the damping ratio ζ. The damping ratio ζ is increased by reducing the gain K.

When $M_p = 0.75$, Let $\zeta = \zeta_1$ and $K = K_1$

When $M_p = 0.25$, Let $\zeta = \zeta_2$ and $K = K_2$

Peak overshoot, $M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

Taking natural logarithm on both sides, $\ln M_p = \frac{-\zeta\pi}{\sqrt{1-\zeta^2}}$

On squaring we get, $(\ln M_p)^2 = \frac{\zeta^2\pi^2}{1-\zeta^2}$

On crossing multiplication we get,

$$(1-\zeta^2)\,(\ln M_p)^2 = \zeta^2\pi^2$$

$$(\ln M_p)^2 - \zeta^2(\ln M_p)^2 = \zeta^2\pi^2$$

$$(\ln M_p)^2 = \zeta^2\pi^2 + \zeta^2(\ln M_p)^2$$

$$(\ln M_p)^2 = \zeta^2\left[\pi^2 + (\ln M_p)^2\right]$$

$$\therefore \zeta^2 = \frac{(\ln M_p)^2}{\pi^2+(\ln M_p)^2} \quad ....(1)$$

$$\text{But } \zeta = \frac{1}{2\sqrt{KT}}, \quad \therefore \zeta^2 = \frac{1}{4KT} \quad ....(2)$$

On equating, equation (1) & (2) we get,

$$\frac{1}{4KT} = \frac{(\ln M_p)^2}{\pi^2+(\ln M_p)^2}$$

$$\frac{1}{K} = \frac{4T\,(\ln M_p)^2}{\pi^2+(\ln M_p)^2}$$

$$K = \frac{\pi^2+(\ln M_p)^2}{4T\,(\ln M_p)^2}$$

When, $K = K_1$, $M_p = 0.75$, $\therefore K_1 = \frac{\pi^2+(\ln 0.75)^2}{4T\,(\ln 0.75)^2} = \frac{9.952}{0.331T} = \frac{30.06}{T}$

When, $K = K_2$, $M_p = 0.25$, $\therefore K_2 = \frac{\pi^2+(\ln 0.25)^2}{4T\,(\ln 0.25)^2} = \frac{11.79}{7.68T} = \frac{1.53}{T}$

$$\therefore \frac{K_1}{K_2} = \frac{(1/T)\,30.06}{(1/T)\,1.53} = 19.6$$

$$K_1 = 19.6\,K_2 \quad \text{(or)} \quad K_2 = \frac{1}{19.6}K_1$$

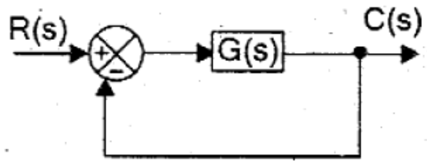
To reduce peak overshoot from 0.75 to 0.25, K should be reduced by 19.6 times (approximately 20 times).

5) A unity feedback control system has an open loop transfer function G(s)=10/s(s+2). Find the rise time, percent overshoot, peak time and settling time for step input of 12 units.

SOL:

The unity feedback system is shown in fig

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$



The closed loop transfer function,

Given that, G(s) = 10/s (s+2)

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{10}{s(s+2)}}{1+\frac{10}{s(s+2)}} = \frac{10}{s(s+2)+10} = \frac{10}{s^2+2s+10}$$

$$\left.\begin{array}{l}\text{Standard form of}\\ \text{Second order transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

On comparing equation (1) & (2) we get,

$$\omega_n^2 = 10 \qquad 2\zeta\omega_n = 2$$

$$\therefore \omega_n = \sqrt{10} = 3.162 \text{ rad/sec} \qquad \therefore \zeta = \frac{2}{2\omega_n} = \frac{1}{3.162} = 0.316$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1}\frac{\sqrt{1-0.316^2}}{0.316} = 1.249 \text{ rad}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.162\sqrt{1-0.316^2} = 3 \text{ rad/sec}$$

$$\text{Rise time, } t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-1.249}{3} = 0.63 \text{ sec}$$

$$\text{Percentage overshoot, } \%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.316\pi}{\sqrt{1-0.316^2}}} \times 100$$

$$= 0.3512 \times 100 = 35.12\%$$

$$\text{Peak overshoot} = \frac{35.12}{100} \times 12 \text{ units} = 4.2144 \text{ units}$$

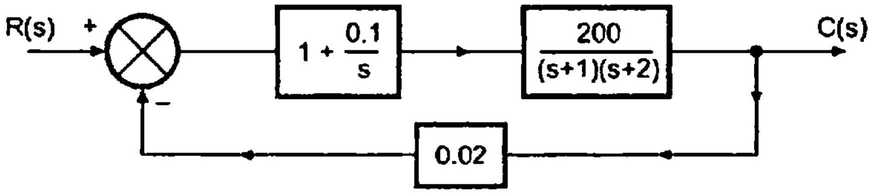
$$\text{Peak time,} \quad t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3} = 1.047 \text{ sec}$$

$$\text{Time constant, } T = \frac{1}{\zeta\omega_n} = \frac{1}{0.316 \times 3.162} = 1 \text{ sec}$$

$\therefore$ For 5% error, Settling time, $t_s = 3T = 3$ sec

For 2% error, Settling time, $t_s = 4T = 4$ sec

6) The control system is shown below. If the input to the system is i) Unit step ii) Unit Ramp. Find ess.



SOL:

$$G(s) = \left(1 + \frac{0.1}{s}\right)\left(\frac{200}{(s+1)(s+2)}\right) = \frac{(s+0.1)200}{s(s+1)\ (s+2)}$$

$$H(s) = 0.2$$

$$G(s)H(s) = \frac{200(s+0.1)}{s(s+1)\ (s+2)} \times 0.02 = \frac{200 \times 0.02 \times 0.1}{1 \times 2} \times \frac{(1+10s)}{s(1+s)\ (1+0.5s)}$$

$$= \frac{0.2(1+10s)}{s(1+s)(1+0.5s)}$$

i) For unit step input

$$K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \frac{0.2(1+10s)}{s(1+s)(1+0.5s)} = \infty$$

$$\therefore \quad e_{ss} = \frac{1}{1+K_p} = 0$$

ii) For unit ramp input

$$K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} \frac{0.2(1+10s)}{s(1+s)(1+0.5s)} = 0.2$$

$$\therefore \quad e_{ss} = \frac{1}{K_v} = \frac{1}{0.2} = 5$$

7) Find the steady state error for various types of standard test inputs for a unity feedback system with a) K=10 b) K=200.

$$G(s) = \frac{K}{s(s+5)\ (s+10)}$$

SOL:

$$G(s)H(s) = \frac{K}{s(s+5)\ (s+10)} = \frac{K}{s \times 5 \times \left(1+\frac{s}{5}\right) \times 10 \times \left(1+\frac{s}{10}\right)}$$

$$= \frac{\left(\frac{K}{50}\right)}{s(1+0.2s)\ (1+0.1s)}$$

$$K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \frac{\left(\frac{K}{50}\right)}{s(1+0.2s)\ (1+0.1s)} = \infty$$

$$K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} \frac{s\left(\frac{K}{50}\right)}{s(1+0.2s)\ (1+0.1s)} = \frac{K}{50}$$

$$K_a = \lim_{s \to 0} s^2G(s)H(s) = \lim_{s \to 0} \frac{s^2\left(\frac{K}{50}\right)}{s(1+0.2s)\ (1+0.1s)} = 0.$$

∴ For step input of magnitude 1,

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0 \qquad \text{... for any value of K.}$$

For ramp input of magnitude 1,

$$e_{ss} = \frac{1}{K_v} = \frac{50}{K}$$

a) For K = 10, $e_{ss} = 5$

b) For K = 200, $e_{ss} = 0.25$

For parabolic input of magnitude 1,

$$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty \qquad \text{... for any value of K.}$$

8) For a unity feedback system G(s) = 20 (s+2) / [s^2 (s+1)(s+5)]. Determine i) type of the system ii) error coefficients iii) steady state error for the input 1 + 3t + t^2/2.

SOL:

(i)

$$G(s)H(s) = \frac{20(s+2)}{s^2(s+1)(s+5)} = \frac{8(1+0.5s)}{s^2(1+s)(1+0.2s)}$$

This is a type 2 system.

(ii)

$$K_p = \underset{s\to 0}{\text{Lt}}\ G(s)H(s) = \underset{s\to 0}{\text{Lt}} \frac{8(1+0.5s)}{s^2(1+s)(1+0.2s)} = \infty$$

$$K_v = \underset{s\to 0}{\text{Lt}}\ sG(s)H(s) = \underset{s\to 0}{\text{Lt}} \frac{8(1+0.5s)}{s(1+s)(1+0.2s)} = \infty$$

$$K_a = \underset{s\to 0}{\text{Lt}}\ s^2G(s)H(s) = \underset{s\to 0}{\text{Lt}} \frac{8(1+0.5s)}{(1+s)(1+0.2s)} = 8$$

(iii) Since the input is non-standard, let us use the formula

$$e_{ss} = \underset{s\to 0}{\text{Lt}} \left[\frac{sR(s)}{1+G(s)H(s)}\right]$$

Given

$$r(t) = 1+3t+\frac{t^2}{2}$$

∴

$$R(s) = \frac{1}{s}+\frac{3}{s^2}+\frac{1}{s^3} = \frac{s^2+3s+1}{s^3}$$

$$e_{ss} = \underset{s\to 0}{\text{Lt}} \frac{sR(s)}{1+G(s)H(s)} = \underset{s\to 0}{\text{Lt}} \frac{(s^2+3s+1)}{s^2+\frac{8(1+0.5s)}{(1+s)(1+0.2s)}}$$

$$= \frac{0+0+1}{0+\frac{8(1+0)}{(1+0)(1+0)}} = \frac{1}{8} = 0.125$$

$$e_{ss} = 0.125$$